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Damped oscillator with quantum noise

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Abstract. We find the general form of positive-energy Gaussian noise so that a linear damped (Bose or Fermi) oscillator obeying a quantum Langevin equation should remain canonical for all time. We show that in a heat bath the system converges to a KMS state as time $t \to \infty$.

1. Introduction and summary

An oscillator at time $t = 0$ can be described by the operators $Q_0 = x$ and $P_0 = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$, acting on $L^2(\mathbb{R})$. An interaction between the oscillator and an external system, such as the heat bath, might be described, at least in principle, by a Hamiltonian $H = \omega A_0^* A_0 + \gamma_I H_I$. Here, $A_0 = 2^{-1/2}(P_0 - \mathrm{i}Q_0)$, γ_I is a positive parameter measuring the strength of the interaction and $\gamma_I H_I$ is the interaction of the oscillator with the external system; H_I involves the dynamical variables of the external system as well as the oscillator, and will be an operator on $L^2(\mathbb{R}) \otimes \Gamma_I$ where Γ_I is the Hilbert space of the external system. If X is an oscillator observable, its expectation in an equilibrium state is then given by

$$\langle X \rangle_\beta = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta H} X)}{\mathrm{Tr}(\mathrm{e}^{-\beta H})} \qquad \beta = \frac{1}{KT}$$

the trace being taken over $L^2(\mathbb{R}) \otimes \Gamma_I$. In practice, H_I is not merely unknown; it involves the detailed interactions of millions of atoms. There have been many attempts to construct a description which avoids the details of H_I and depends only on γ_I and the temperature, when the external system is a heat bath. One such method is to use the concept of quantum noise. This leads to a non-Hamiltonian description of the oscillator. The quantum noise is described by quantised fields $\phi(t)$, $\pi(t)$, $t \geqslant 0$, acting on a Hilbert space Γ, often related to Γ_I. The noise acts as the driving term of the damped oscillator equations

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \omega P(t) - \gamma Q(t) + \phi(t) \qquad \frac{\mathrm{d}P(t)}{\mathrm{d}t} = -\omega Q(t) - \gamma P(t) + \pi(t) \qquad (1)$$

with initial conditions $Q(0) = Q_0 \otimes I$, $P(0) = P_0 \otimes I$. The solutions $Q(t)$, $P(t)$ are operators on $L^2(\mathbb{R}) \otimes \Gamma$. For simplicity we have written $\phi(t)$ for $I \otimes \phi(t)$ and $\pi(t)$ for $I \otimes \pi(t)$, and we understand (1) to be the unsmeared forms of distribution equations. These equations are simple linear quantum analogues of Langevin equations. Or, one can motivate the choice of these equations, as described by Haken (1970) by showing that they follow from the full Hamiltonian equations by a series of plausible approximations. We shall simply take them as our starting point.

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Various papers, surveyed by Haken (1970) show that the decay parameter γ is proportional to γ_I; they also arrive at a formula for $\phi(t)$, $\pi(t)$ in terms of the interaction. At this point, simplifying assumptions are made. Let us summarise the arguments motivating Senitzky (1960). If the noise, $\phi(t)$, $\pi(t)$ is omitted, the system is exactly solvable, and the motion takes place entirely in the first factor, $L^2(\mathbb{R})$ of $L^2(\mathbb{R})\otimes\Gamma$. The solution then is

$$Q_1(t) = (Q_0 \cos \omega t + P_0 \sin \omega t)\, e^{-\gamma t} \qquad P_1(t) = (-Q_0 \sin \omega t + P_0 \cos \omega t)\, e^{-\gamma t}. \tag{2}$$

From this we see that the commutator decays away

$$[Q_1(t), P_1(t)] = i\, e^{-2\gamma t} \tag{3}$$

and the model violates the uncertainty relations. Senitzky (1960) asked, can we choose (non-commuting) noise operators such that the canonical commutation relations hold for all time? He found an approximate solution to his question, if $\gamma \ll \omega$, by choosing $\phi(t)$ to be the positive-energy part of white noise. Lax (1965) makes a different choice, namely the whole spectrum of quantum white noise, obeying

$$[\phi(t), \pi(t')] = i\lambda\delta(t-t'). \tag{4}$$

He found that by a suitable choice of λ he could exactly compensate for the decay (3) by the term coming from the noise. He obtained a relation between γ and λ, a quantum 'fluctuation dissipation theorem'. Since Lax needs the full range of energies, $-\infty < k < \infty$, to construct quantum white noise, he introduces negative-energy states into the theory. The model is therefore deficient, as noted by Kubo (1969) and Haken (1970). A simple consequence of the negative-energy states is that the 'equilibrium-state' at $t=\infty$ fails to satisfy the KMS condition, as we shall see. As explained by Kubo (1957), Martin and Schwinger (1959) and Haag *et al* (1967), the KMS condition is satisfied by any realistic theory of equilibrium states as a consequence of the positive-energy axiom.

In this paper we show that there is a choice of $\phi(t)$, $\pi(t)$ without negative-energy states which also solves the commutator problem exactly. We find (§ 2) the complete set of models of this type; each model is defined by a function $\sigma(k)$, defined for $0 \le k \le \omega$, satisfying $0 \le \sigma(k) \le 2\gamma/\pi$. We show in § 3 that if the noise is in a Gibbs state, then as $t \to \infty$ the models converge to a KMS state at the same temperature. The equilibrium state exhibits line broadening of width γ and is determined by β and $\sigma(k)$.

The theory is extended to fermions and to spins (§ 4) and to systems with asymmetric damping (§ 5).

2. The quantum fluctuation–dissipation theorem

Let $A(t) = 2^{-1/2}(P(t) - iQ(t))$ and $a(t) = 2^{-1/2}(\pi(t) - i\phi(t))$. Then equation (1) separates into

$$\frac{dA(t)}{dt} = (-i\omega - \gamma)A(t) + a(t)$$

and its Hermitian conjugate. The solution to this, subject to $A(0) = A_0$, is

$$A(t) = \exp((-i\omega - \gamma)t)A_0 + \int_0^t ds\, \exp[(-i\omega-\gamma)(t-s)]a(s). \tag{5}$$

In this paper we assume that $a(t)$ has a Fourier decomposition

$$a(t)=\int_0^\infty \rho(k)\,\mathrm{e}^{-\mathrm{i}kt}a(k)\,\mathrm{d}k \tag{6}$$

where $a(k)$, $k\geqslant 0$ are oscillators, so that

$$[a(k),\,a^*(k')]=\delta(k-k'). \tag{7}$$

We take ρ to be non-negative. The choice of $k=0$ as the lower limit of the energy integral in (6) expresses the need not to add quanta of negative energy to A^*.

The problem is to find the most general ρ such that $[A(t),\,A^*(t)]=1$ for all $t\geqslant 0$. We note that $[A_0, A_0^*]=1$ and using (5), (6) and (7) we obtain

$$\begin{aligned}1&=[A(t),\,A^*(t)]\\ &=\mathrm{e}^{-2\gamma t}+\int_0^\infty \mathrm{d}k\,\rho^2\int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\,\exp[(-\mathrm{i}\omega-\gamma)(t-s)\\ &\quad -\mathrm{i}ks+(\mathrm{i}\omega-\gamma)(t-s')+\mathrm{i}ks']\end{aligned}$$

so

$$1=\mathrm{e}^{-2\gamma t}\left(1+\int_0^\infty \rho^2 G\,\mathrm{d}k\{(\mathrm{e}^{2\gamma t}+1-2\mathrm{e}^{\gamma t}\cos[(\omega-k)t]\}\right) \tag{8}$$

where $G(k)=[\gamma^2+(\omega-k)^2]^{-1}$. Comparing as $t\to\infty$ gives

$$\int \mathrm{d}k\,\rho^2(k)G(k)=1. \tag{9}$$

We see from (8) and (9) that for all $t\geqslant 0$,

$$1+\mathrm{e}^{2\gamma t}+1-2\mathrm{e}^{\gamma t}\int_0^\infty \mathrm{d}k\,\rho^2 G\cos[(\omega-k)t]=\exp(2\gamma t).$$

Therefore we seek the most general solution of the integral equation for $\rho(k)$

$$\int_0^\infty \mathrm{d}k\,\rho^2(k)\cos[(\omega-k)t][\gamma^2+(\omega-k)^2]^{-1}=\exp(-\gamma t) \qquad t\geqslant 0.$$

Hence

$$\int_{-\omega}^\infty \mathrm{d}k\,\rho^2(k+\omega)\cos(kt)(\gamma^2+k^2)^{-1}=\mathrm{e}^{-\gamma t}.$$

This gives the cosine transform

$$\int_0^\infty \mathrm{d}k[\rho^2(k+\omega)+\rho^2(-k+\omega)](\gamma^2+k^2)^{-1}\cos(kt)=\mathrm{e}^{-\gamma t} \qquad t\geqslant 0. \tag{10}$$

A simple contour integral shows

$$\int_0^\infty \mathrm{d}k\,\frac{2\gamma}{\pi}\,\frac{\cos(kt)}{(\gamma^2+k^2)}=\mathrm{e}^{-\gamma t} \qquad t\geqslant 0.$$

Since a cosine transform has a unique inverse, we must have

$$\rho^2(k+\omega)+\rho^2(-k+\omega)=2\gamma/\pi.$$

If $\rho(k)$ could be non-zero for $k<0$, a possible solution would be

$$\rho(k)=\gamma/\pi \qquad \text{for all } k\in\mathbb{R}.$$

This is essentially the choice of Lax (1965). But since we seek solutions with positive energy, we impose the condition $\rho(k)=0$, $k<0$. We now see immediately that the most general solution to the commutator problem is described as follows: let $\sigma(k)$, $0\leqslant k\leqslant\omega$ be any measurable function satisfying $0\leqslant\sigma(k)\leqslant 2\gamma/\pi$. Then $\rho(k)$ is determined by

$$\rho^2(k)=\begin{cases} 0 & k<0 \\ \sigma(\omega-k) & 0\leqslant k<\omega \\ 2\gamma/\pi-\sigma(k-\omega) & \omega\leqslant k<2\omega \\ 2\gamma/\pi & k\geqslant 2\omega. \end{cases} \tag{11}$$

We see that the noise depends on the system, in that ω appears in (11). We should therefore regard $\phi(t)$, $\pi(t)$ as the influence of the noise on the system, rather than as heat-bath variables as such. We shall see (§ 3) that $\sigma(k)$ is determined by the lineshape in equilibrium, in the energy range $0\leqslant k\leqslant\omega$.

Whatever the choice of $\sigma(k)$, the noise is not local in time; thus $[\phi(t), \pi(t')]\neq \mathrm{i}\delta(t-t')$ and even the field does not commute with itself at different times. This contrasts with the very simple properties of the Lax model, in which the noise is time local and strongly Markovian.

The solutions $(Q(t), P(t))$, operators in $L^2(\mathbb{R})\otimes\Gamma$ for each time, are not really time-displacement invariant. For, the initial choice $Q(0)=Q_0$, $P(0)=P_0$, commute with all future noise; but the solution $Q(t_0)$, $P(t_0)$, $t_0>0$ does not commute with all noise future to t_0. In this sense, the initial choice, (Q_0, P_0) was the wrong choice of canonical pair for the Hamiltonian $\omega A_0^*A_0+\gamma_1 H_1$, being the correct one for $\omega A_0^*A_0$. The solution, then, corresponds rather better to the time-dependent Hamiltonian $\omega A_0^*A_0+\theta(t)\gamma_1 H_1$, where θ is the Heaviside function. The complicated time dependence of $(Q(t), P(t))$ describes the system's attempt to adjust to the new Hamiltonian for $t>0$. Physically, it should describe an oscillator shot into a hot environment at $t=0$. We now show that for a natural choice of state of the noise, the large-time limit of the theory exists and defines a stochastic process with an invariant state.

3. The approach to equilibrium

The advantage of insisting that $[Q(t), P(t)]=\mathrm{i}\hbar$ exactly for all time, rather than just approximately, is that these operators define a Weyl system at time t:

$$W_t(\alpha,\beta)=\exp \mathrm{i}(\alpha Q(t)+\beta P(t)) \qquad \alpha,\beta\in\mathbb{R}.$$

These generate the canonical c^* algebra, $\mathcal{A}_t$, of operators on $L^2(\mathbb{R})\otimes\Gamma$. All the algebras $\mathcal{A}_t$ are isomorphic, $t\geqslant 0$, to $\mathcal{A}_0$. The theory thus obeys the axioms of Accardi *et al* (1981, see also Accardi 1981, Lewis 1981). Suppose Γ is spanned by vectors obtained by acting on a time invariant cyclic vector Ω by the noise operators $\phi(t)$, $\pi(t)$ (smeared with suitable test functions). By 'time invariant' we mean that there is a unitary motion, $U_\Gamma(t)$ acting on Γ, such that

$$U_\Gamma(t)\phi(s)U_\Gamma^{-1}(t)=\phi(s+t) \qquad U_\Gamma(t)\pi(s)U_\Gamma^{-1}(t)=\pi(s+t)$$

and

$$U_\Gamma(t)\Omega = \Omega.$$

Let us now regard states as positive linear forms, rather than vectors. Then for each state $\tilde{\omega}$, pure or not, on the initial Weyl system W_0, we define a state $\tilde{\omega}_t$ on W_0 by

$$\tilde{\omega}_t(W_0(\alpha, \beta)) = (\tilde{\omega} \otimes \Omega)(W_t(\alpha, \beta)). \tag{12}$$

The motion, $\tilde{\omega} \to \tilde{\omega}_t$, through the state space of the abstract algebra $\mathscr{A}_0$, is the time evolution predicted by our theory.

We see from (5) that $W_t(\alpha, \beta)$ factorises into the system $W_{1t}(\alpha, \beta)$ coming from (2)

$$W_{1t}(\alpha, \beta) = \exp \mathrm{i}(\alpha Q_1(t) + \beta P_1(t))$$

and a factor X_t coming from the noise. Thus

$$\tilde{\omega}_t(W_0(\alpha, \beta)) = \tilde{\omega}(W_{1t}(\alpha, \beta))\Omega(X_t(\alpha, \beta)).$$

According to (2), $\tilde{\omega}(W_{1t}(\alpha, \beta))$ converges to 1 as $t \to \infty$, and the system forgets its initial state exponentially fast. To proceed, for example to prove that $\tilde{\omega}_t$ converges as $t \to \infty$, we make assumptions about the noise state Ω. Suppose that the noise is Gaussian, and that the oscillators $a^*(k)$ are all independent; then

$$\langle a^*(k)a(k')\rangle_\Omega = n(k)\delta(k-k') \qquad \langle a(k')a^*(k)\rangle_\Omega = (n(k)+1)\delta(k-k') \tag{13}$$

with $n \geq 0$. Suppose also

$$\langle a(k)a(k')\rangle_\Omega = 0 \tag{14}$$

which states that Ω is gauge invariant. Then the limit of the state $\tilde{\omega}_t$, given by (12), as $t \to \infty$, if it exists, will be Gaussian, and will therefore be determined by the two-point functions

$$\begin{aligned}
&\langle A^*(t)A(t+\tau)\rangle \\
&\quad = \tilde{\omega}(A_0^*A_0)\exp(-\mathrm{i}\omega\tau - \gamma\tau - 2\gamma t) + \int_0^\infty \mathrm{d}k\, n(k)\rho^2(k) \int_0^t \mathrm{d}s \int_0^{t+\tau} \mathrm{d}s' \\
&\quad\quad \times \exp[(\mathrm{i}\omega - \gamma)(t-s) + \mathrm{i}ks + (-\mathrm{i}\omega - \gamma)(t+\tau-s') - \mathrm{i}ks'].
\end{aligned}$$

As $t \to \infty$ this converges to

$$\langle A_\infty^* A_\infty(\tau)\rangle = \int_0^\infty \mathrm{d}k\, n(k)\rho^2(k)G(k)\,\mathrm{e}^{-\mathrm{i}k\tau}. \tag{15}$$

Similarly, $\langle A(t+\tau)A^*(t)\rangle$ converges to

$$\langle A_\infty(\tau)A_\infty^*\rangle = \int_0^\infty \mathrm{d}k(n(k)+1)\rho^2(k)G(k)\,\mathrm{e}^{-\mathrm{i}k\tau}. \tag{16}$$

The KMS condition states that at a temperature $\theta = \beta^{-1}$ we have

$$\langle A_\infty^* A_\infty(\tau + \mathrm{i}\beta)\rangle = \langle A_\infty(\tau)A_\infty^*\rangle. \tag{17}$$

It is here that the positive-energy condition $k \geq 0$ comes in. It ensures that the integral in (15) converges exponentially and defines an analytic function in the strip $\{\tau \in \mathbb{C}: 0 < \operatorname{Im}\tau < \beta\}$, provided $n(k) = (\mathrm{e}^{\beta k} - 1)^{-1}$. For if $k \geq 0$, the factor $(\mathrm{e}^{\beta k} - 1)^{-1}$ provides a convergence factor as $k \to \infty$ which dominates $\mathrm{e}^{\operatorname{Im}\tau k}$ if $\operatorname{Im}\tau < \beta$. This choice of $n(k)$ also ensures that (17) holds. We also remark that (14) ensures that $\langle A_\infty A_\infty(\tau)\rangle = 0$, which obviously is an analytic function so the KMS condition is trivially true. Equations (13) and (14), with $n(k)=(e^{\beta k}-1)^{-1}$, may be expressed by saying that Ω is a KMS state for the time evolution of the oscillators $a^*(t)$, $a(t)$.

The limit (15) and (16) defines a process $P_\infty(\tau)$, $Q_\infty(\tau)$ which form a canonical pair for each $\tau \in \mathbb{R}$. These operators act on a Hilbert space (not $L^2(\mathbb{R})\otimes\Gamma$) reconstructed from the quasi-free Wightman functions obtained from (15) and (16) according to the well known method (Wightman 1956). The requisite positivity conditions hold, as the functions (15) and (16) are limits of matrix elements of operators on $L^2(\mathbb{R})\otimes\Gamma$. The process they define is stationary, canonical, Gaussian and KMS, and is similar to those studied by Lewis and Thomas (1975), and called FKM processes, after Ford *et al* (1965).

As $\gamma\to 0$ the limit process $A_\infty(\tau)$, $A^*_\infty(\tau)$, converges to a free oscillator at temperature θ, for any smooth σ. The limit process describes decaying canonical quantum systems for which the decay mechanism has always been present. At zero temperature, $Q_\infty(\tau)$ is the non-relativistic analogue of the generalised free field, used by Matthews and Salam (1958, 1959) to describe unstable elementary particles. Our theory supports their ideas and gives an extension to non-zero temperatures. It would be interesting to search for a temperature dependence in the decay curves of elementary particles, as predicted by the relativistic version of (15) and (16). This could answer the question, does the virtual quantised field that causes a decay take on the temperature of the surroundings? This would be detectable at temperatures of the order of magnitude as ω, which is the Q value of the decay. Thus the process $\pi^0\to 2\gamma$ should be influenced by the heat bath well below a million degrees K. At such temperatures we could test whether the pion is described by a canonical field.

The process at $t=\infty$, given by (15), (16) is parametrised by the temperature $1/\beta$, the decay width γ (which represents the strength of the interaction), and the function $\sigma(k)$, which is all that is left of the details of H_I. The lineshape of the system, given by the Fourier transform of (15), is directly parametrised by $\sigma(k)$.

4. The fermion and spin models

Consider creation and annihilation operators B^* and B obeying anticommutation relations $\{B^*, B\}_+ = I$, and consider the stochastic equation

$$\frac{dB^*(t)}{dt} = i\omega B^*(t) - \gamma B^*(t) + b^*(t) \tag{18}$$

subject to the condition $B^*(0)=B^*$; here

$$b^*(t) = \int_0^\infty \rho(k) b^*(k) \exp(ikt)\, dk \tag{19}$$

and the $b^*(k)$ are fermion operators obeying $\{b^*(k), b(k')\}_+ = \delta(k-k')$. The fermions are in a representation based on a quasi-free state Ω in a Hilbert space $\hat{\Gamma}$. The operators B, B^* act on $\mathbb{C}^2$, and the solution

$$B^*(t) = B^*\, e^{(i\omega-\gamma)t} + \int_0^t ds[\exp(i\omega-\gamma)(t-s)]b^*(s) \tag{20}$$

takes place in $\mathbb{C}^2\otimes\hat{\Gamma}$. Here as before we use the shorthand B^* for $B^*\otimes I$ and $b^*(s)$

for $I \otimes b^*(s)$. As usual, we require the canonical relations $B^*(t)B(t)+B(t)B^*(t)=I$ for all $t \geqslant 0$. This leads to equation (8) for ρ and to its solution equation (11). This would model a two-level atom, $\mathbb{C}^2$ describing the two states, and ω being the energy difference. For fermions we postulate the Gaussian state

$$\langle b^*(k)b(k')\rangle_\Omega = n(k)\delta(k-k') \tag{21}$$

and as before the limit state at $t=\infty$ is a KMS state if and only if

$$n(k) = (\mathrm{e}^{\beta k}+1)^{-1}. \tag{22}$$

The infinite-time state is then the quasi-free state with the two-point function

$$\langle B^*_\infty B_\infty(\tau)\rangle = \int_0^\infty \mathrm{d}k\, n(k)\rho^2(k)G(k)\exp(-\mathrm{i}k\tau) \tag{23}$$

which defines a Hilbert space and operators $B^*_\infty(t)$, $t \in R$, by the Wightman reconstruction theorem. The resulting process is stationary and KMS.

The operators $B(t)$, $B^*(t)$ can be used to build a model of the relaxation of a particle of spin $\frac{1}{2}$ in a magnetic field. To do this, define

$$\sigma_1(t) = B^*(t)+B(t) \qquad \sigma_2(t) = \mathrm{i}(B^*(t)-B(t)) \qquad \sigma_3(t) = 1-2B^*(t)B(t). \tag{24}$$

The Hamiltonian $H=-h\sigma_3$ describes a spin in a magnetic field h in the third direction. This leads to the Fermi oscillator with $\omega = 2h$. In this model the spin is subject to anticommuting noise; this has the virtue that the spin-$\frac{1}{2}$ relations

$$\sigma_i\sigma_j = \varepsilon_{ijk}\sigma_k \qquad \sigma_j^2 = I \tag{25}$$

hold identically at all times. This would not appear to be true for the model described by Agarwal (1974). Our model makes predictions about the lineshape and time behaviour of spin relaxation. Let us define the line density $N(k)$ in equilibrium by

$$\langle B^*_\infty B_\infty(t)\rangle = \int_0^\infty \mathrm{d}k\, N(k)\exp(-\mathrm{i}tk).$$

Then equation (11) requires that for $0<x<\omega$,

$$(\mathrm{e}^{\beta(\omega-x)}+1)N(\omega-x)+(\mathrm{e}^{\beta(\omega+x)}+1)N(\omega+X) = (2\gamma/\pi)(\gamma^2+x^2)^{-1}. \tag{26}$$

One shows that if $N(k)$ is continuous and the left and right limits $N'(\omega-)$ and $N'(\omega+)$ exist, then $N(k)$ is differentiable at ω and $N'(\omega)=0$. Then $N(k)$ has a maximum at $k=\omega$, and there is no shift in the position of the line. From (26) if $N'(\omega)=0$ we obtain by differentiation

$$N''(\omega+)+N''(\omega-) = -2[(2\gamma^2+\beta^2)\,\mathrm{e}^{\beta\omega}+2\gamma^2](\mathrm{e}^{\beta\omega}+1)^{-2}/(\gamma\pi)$$

for all β. This is a good test of our theory, as $\sigma(k)$ has dropped out. Naturally, another good test is

$$N(k) = 2\gamma(\mathrm{e}^{\beta k}+1)^{-1}(\gamma^2+(\omega-k)^2)^{-1}/\pi$$

for all β and all $k>2\omega$.

5. Asymmetric damping

Consider now the case where the interaction with the heat bath causes $P(t)$ and $Q(t)$ to undergo unequal damping:

$$\frac{dQ}{dt} = \omega P - \gamma_1 Q \qquad \frac{dP}{dt} = -\omega Q - \gamma_2 P.$$

Let $\omega' = (\omega^2 - (\gamma_1 - \gamma_2)^2/4)^{1/2}$ and take the case of undercritical damping, i.e. ω' is real. The eigenvalues of the system are given by λ:

$$\det\begin{vmatrix} -\gamma_1 - \lambda, & \omega \\ -\omega, & -\gamma_2 - \lambda \end{vmatrix} = \lambda^2 + (\gamma_1 + \gamma_2)\lambda + (\gamma_1\gamma_2 + \omega^2) = 0.$$

Thus $\lambda_1 = -(\gamma_1 + \gamma_2)/2 + i\omega'$, $\lambda_2 = \overline{\lambda_1}$. Let $A = [Q - (\lambda_2 + \gamma_1)P/\omega]/i\sqrt{2}$. Then $A^* = -[Q - (\lambda_1 + \gamma_1)P/\omega]/i\sqrt{2}$ and

$$[A, A^*] = \tfrac{1}{2}[-i(\lambda_1 + \gamma_1)/\omega + i(\lambda_2 + \gamma_1)/\omega] = i(\lambda_2 - \lambda_1)/2\omega = \omega'/\omega.$$

Then

$$\begin{aligned} dA/dt &= [dQ/dt - (\lambda_2 + \gamma_1)\omega^{-1}\, dP/dt]/i\sqrt{2} \\ &= [\omega P - \gamma_1 Q - (\lambda_2 + \gamma_1)(-\omega Q - \gamma_2 P)/\omega]/i\sqrt{2} \\ &= [\lambda_2 Q + (\omega^2 + \lambda_2\gamma_2 + \gamma_1\gamma_2)P/\omega]/i\sqrt{2}. \end{aligned}$$

Now, $\omega^2 + \lambda_2\gamma_2 + \gamma_1\gamma_2 = -\lambda_2\gamma_1 - \lambda_2^2$. Hence $dA/dt = \lambda_2(Q - (\lambda_2 + \gamma_1)P/\omega)/i\sqrt{2} = \lambda_2 A$. Hence also $dA^*/dt = \bar{\lambda}_2 A^* = \lambda_1 A^*$.

In the corresponding stochastic equations we must add positive energy quanta to A^*, so we try

$$\frac{dA^*}{dt} = \lambda_1 A^* + a^*(t) \qquad \text{where } a^*(t) = \int_0^\infty dk\, \rho(k) a^*(k) \exp(ikt).$$

The solution is

$$A^*(t) = \exp(\lambda_1 t)A^* + \int_0^t ds\, \exp[\lambda_1(t-s)]a^*(s)$$

$$A(t) = \exp(\lambda_2 t)A + \int_0^t ds'\, \exp[\lambda_2(t-s)]a(s').$$

The commutator must be ω'/ω for all t. Thus

$$\begin{aligned} \omega'/\omega &= [A(t), A^*(t)] \\ &= \frac{\omega'}{\omega}\exp[(\lambda_1 + \lambda_2)t] + \int_0^\infty dk\, \rho^2(k) \exp[(\lambda_1 + \lambda_2)t] \\ &\quad \times \{\exp[(-\lambda_1 + ik)t] - 1\}\{\exp[(-\lambda_2 - ik)t] - 1\}/[(-\lambda_1 + ik)(-\lambda_2 - ik)] \\ &= \exp(-2\gamma t)\left(\frac{\omega'}{\omega} + \int_0^\infty dk\, \rho^2(k)\{e^{2\gamma t} + 1 - 2e^{\gamma t}\cos[t(\omega' - k)]\}G(\gamma, \omega'; k)\right) \end{aligned}$$

where $2\gamma = -(\lambda_1 + \lambda_2) = \gamma_1 + \gamma_2$.

This leads to the same problem as in § 2, with ω' replacing ω in (8) and ω'/ω replacing 1. We obtain the usual shift in the spectrum, as well as the broadening. It does not seem possible to make sense of stochastic equations that are critically or over critically damped.

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